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Robust stability analysis of uncertain systems with two additive time-varying delay components

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ABSTRACT

This paper is concerned with stability analysis for uncertain systems. The systems are based on a new time-delay model proposed recently, which contains multiple successive delay components in the state. The relationship between the time-varying delay and its upper bound is taken into account when estimating the upper bound of the derivative of Lyapunov functional. As a result, some less conservative stability criteria are established for systems with two successive delay components and parameter uncertainties. Numerical examples show that the proposed criteria are effective and are an improvement over some existing results in the literature.

1. Introduction

It is well known that time-delay is a common phenomenon in many industrial and engineering systems, such as manufacturing systems, telecommunication and economic systems, and is one of the instability sources for dynamical systems. Since system stability is an essential requirement in many applications, the stability problems for time-delay systems have been extensively studied in recent years [1–20]. The robust stability of uncertain systems with interval time-varying delay is investigated in [1] by introducing of uncorrelated augmented matrix items into the Lyapunov functional and using a tighter bounding technology. In [2], the authors study the networked $H\infty$ stabilization of linear time-invariant systems under quantized state feedback control. In [6], the stability analysis for systems with time-varying delay in a range is considered and some delay-range-dependent stability criteria are derived based on the consideration of range for the time-delay. The problem of delay-dependent robust stability for uncertain stochastic systems and their corresponding deterministic systems with time-varying delay are studied in [16]. In [18], the authors first introduce the notion of exponential convergence to a ball containing the origin of the state space. Then, a specific class of uncertain systems is considered and controllers are presented which assure convergence at a rate which is independent of the uncertainty and is the same as that of a nominal system. Finally, another class of uncertain systems is considered and the rate of exponential convergence can be made arbitrarily large. In [5,7–15], the authors discuss the uncertain systems and some stability criteria are proposed. Almost all the reported results on time-delay systems are based on the following basic mathematical model.

$$\dot{x}(t) = Ax(t) + Bx(t - d(t))$$

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where $d(t)$ is a time delay in the state $x(t)$, which is often assumed to be either constant or time-varying satisfying certain conditions, e.g.,

$$0 \leqslant d(t) \leqslant \bar{d} < \infty, \quad \dot{d}(t) \leqslant \tau < \infty.$$

It is worth mentioning that in this model, the time delay in the state variable $x(t)$ is assumed to appear in a singular or simple form. As mentioned in [21], sometimes in practical situations, however, signals transmitted from one point to another may experience a few segments of networks, which can possibly induce successive delays with different properties due to the variable network transmission conditions. Fig. 1. shows one simple example of such situation, which can be found in the networked control system.

In Fig. 1, there are two delays: $d_s(t)$ is used to represent the delay from sensor to controller and $d_a(t)$ is used to represent the delay from controller to actuator. Since the properties of these two delays may not be identical due to the network transmission conditions, it is not reasonable to combine them together. Therefore, when the physical plant and state-feedback controller are, respectively, given by $\dot{x}(t) = Ax(t) + Bu(t)$ and $u_c(t) = Kx_c(t)$, the closed-loop system is given by

$$\dot{x}(t) = Ax(t) + BKx(t - d_s(t) - d_a(t)).$$

Thus, in [21], the following new model for time-delay systems is proposed:

$$\dot{x}(t) = Ax(t) + Bx\left(t - \sum_{i=2}^{n} d_i(t)\right).$$

This model contains multiple delay components in the state and a stability analysis result is reported in [21] for systems with two successive delay components. A numerical example shows the advantage of the stability result. Very recently, in [22], the improved stability criteria are proposed by defining a new Lyapunov functional.

It is worth noting that the stability criteria in [22,21] leave much room for improvement. A significant source of conservativeness that could be further reduced lies in the calculation of the time-derivative of the Lyapunov functional. For example, in [21], the derivative of $\int_{-\bar{d}_1}^{0}\int_{\beta}^{0}\dot{x}^T(t+\alpha)M_1\dot{x}(t+\alpha)d\alpha d\beta$ and $\int_{-\bar{d}_1-\bar{d}_2}^{-\bar{d}_1}\int_{\beta}^{0}\dot{x}^T(t+\alpha)M_2\dot{x}(t+\alpha)d\alpha d\beta$ are estimated as $\bar{d}_1\dot{x}^T(t)M_1\dot{x}(t) - \int_{t-d_1(t)}^{t}\dot{x}^T(\alpha)M_1\dot{x}(\alpha)d\alpha$ and $\bar{d}_2\dot{x}^T(t)M_2\dot{x}(t) - \int_{t-d_1(t)-d_2(t)}^{t-d_1(t)}\dot{x}^T(\alpha)M_2\dot{x}(\alpha)d\alpha$, respectively, which may lead to considerable conservativeness. In [22], the delay term $d_1(t)$ with $0 \leqslant d_1(t) \leqslant \bar{d}_1$ is enlarged as $\bar{d}_1$ and another term $d(t) - d_1(t)$ is also regarded as $\bar{d} - \bar{d}_1$. Moreover, the term $\bar{d} - d(t)$ with $0 \leqslant d(t) \leqslant \bar{d}$ is enlarged as $\bar{d}$. That is, $\bar{d} = d_1(t) + (d(t) - d_1(t)) + (\bar{d} - d(t))$ is enlarged as $2\bar{d}$. So, the aforementioned treatment may lead to a conservative result. Furthermore, in [22,21], the parameter uncertainties are not taken into account.

Motivated by the results of [22,21], it is our intention in this paper to present new stability criteria for uncertain systems with multiple successive delay components. The systems may contain both time-varying system parameter uncertainties and multiple successive delay components. As mentioned in [22], we still consider the case where only two successive delay components appear in the state, and the idea behind this paper can be easily extended to systems with multiple successive

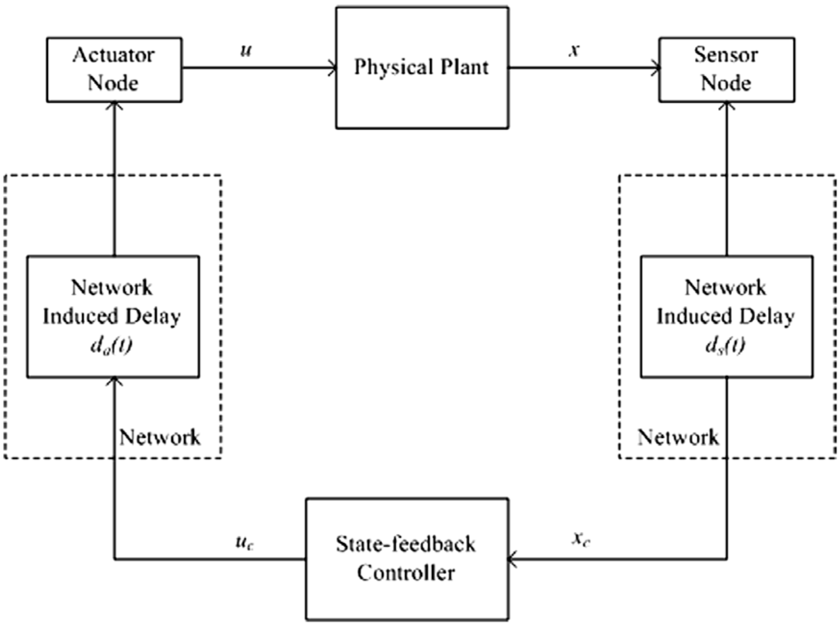

Fig. 1. Networked control system.

delay components. In Section 2, problem formulation is formulated and related preliminaries are presented. In Section 3, by considering the relationship between the time-varying delay and its upper bound when estimating the upper bound of the derivative of Lyapunov functional [23], some less conservative stability criteria are proposed to guarantee the systems with two successive delay components to be robustly asymptotically stable for all admissible parameter uncertainties. These criteria are expressed as a set of linear matrix inequalities (LMIs), which can be readily solved by using standard numerical software [24]. In Section 4, numerical examples are given to illustrate the effectiveness and benefits of the proposed method.

Notations: The notations used throughout the paper are fairly standard. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrix. In symmetric block matrices or long matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem formulation and some preliminaries

Consider the following uncertain continuous system with two additive time-varying delay components:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)x(t - d_1(t) - d_2(t)), \\ x(t) &= \varphi(t) \quad t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_t(t)]^T$ is the state vector, $d_1(t)$ and $d_2(t)$ represent the two delay components in the state and we denote $d(t) = d_1(t) + d_2(t)$; $A$and B are system matrices with appropriate dimensions.

In order to obtain our main results, the assumptions are always made throughout this paper.

Assumption 1. The time-varying delays $d_1(t)$ and $d_2(t)$ satisfy

$$0 \leqslant d_1(t) \leqslant h_1 < \infty, \quad 0 \leqslant d_2(t) \leqslant h_2 < \infty, \tag{2}$$

$$\dot{d}_1(t) \leqslant \tau_1 < \infty, \quad \dot{d}_2(t) \leqslant \tau_2 < \infty, \tag{3}$$

where h_1, h_2, τ_1 and τ_2 are positive constants.

Assumption 2. The parameter uncertainties ΔA,ΔB are of the form:

$$[\,\Delta A \quad \Delta B\,] = HF(t)[\,E_a \quad E_b\,], \tag{4}$$

in which H, E_a, E_b are known constant matrices with appropriate dimensions. The uncertain matrix $F(t)$ satisfies

$$F^T(t)F(t) \leqslant I \quad \text{for} \quad \forall t \in \mathrm{R}. \tag{5}$$

The purpose of this subsection is to derive new stability conditions under which system (1) is asymptotically stable for all delays $d_1\,(t)$ and $d_2(t)$ satisfying (2) and (3). One possible approach to check the stability of this system is to simply combine $d_1(t)$ and $d_2(t)$ into one delay $d(t)$ with

$$0 \leqslant d(t) \leqslant h < \infty, \tag{6}$$

$$\dot{d}(t) \leqslant \tau < \infty, \tag{7}$$

where $h = h_1 + h_2$, $\tau = \tau_1 + \tau_2$.

Then, system (1) becomes

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)x(t - d(t)), \\ x(t) &= \varphi(t) \quad t \in [-h, 0]. \end{aligned} \tag{8}$$

By using some existing stability conditions, the stability of system (8) can be readily checked. As discussed in [22,21], however, since this approach does not make full use of the information on $d_1(t)$ and $d_2(t)$, it would be inevitably conservative for some situations.

In the following, we will develop some practically computable stability criteria for the system described by (1)–(4). The following lemmas are useful in deriving the criteria.

Lemma 1 (Schur complement). *Given constant symmetric matrices $\sum_1, \sum_2$ and $\sum_3$, where $\sum_1 = \sum_1^T$ and $0 < \sum_2 = \sum_2^T$, then $\sum_1 + \sum_3^T \sum_2^{-1} \sum_3 < 0$ if and only if*

$$\begin{bmatrix} \sum_1 & \sum_3^T \\ \sum_3 & -\sum_2 \end{bmatrix} < 0, \quad or \quad \begin{bmatrix} -\sum_2 & \sum_3 \\ \sum_3^T & \sum_1 \end{bmatrix} < 0.$$

Lemma 2 ([25]). *For any $z,y \in R^{n\times m}$ and a positive scalar ε, the following inequality:*

$$2z^Ty \leqslant \varepsilon z^Tz + \varepsilon^{-1}y^Ty$$

holds.

3. Main results

In order to discuss robust stability of system (1), which has parametric uncertainties (4), first, we consider the case in which the matrices A and B are fixed, i.e., $\Delta A = 0$ and $\Delta B = 0$. For this case, the following theorem holds.

Theorem 1. *For given scalars $h_1 > 0$, $h_2 > 0$, τ_1 and τ, the system (1) with delays $d_1(t)$ and $d_2(t)$ is asymptotically stable, if there exist matrices $P > 0, Q_1 \geqslant Q_2 \geqslant 0, R_1 \geqslant R_2 \geqslant 0, Z_1 \geqslant Z_2 > 0, Z_3 > 0, X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \star & X_{22} & X_{23} \\ \star & \star & X_{33} \end{bmatrix} \geqslant 0, Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ \star & Y_{22} & Y_{23} \\ \star & \star & Y_{33} \end{bmatrix} \geqslant 0,\quad D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \star & D_{22} & D_{23} \\ \star & \star & D_{33} \end{bmatrix} \geqslant 0, K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}, L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$, and $W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$, such that the following LMIs (9)–(14) hold:*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & -M_1 - N_1 & -W_1 & A^TU \\ \star & \Psi_{22} & \Psi_{23} & -M_2 - N_2 & -W_2 & 0 \\ \star & \star & \Psi_{33} & -M_3 - N_3 & -W_3 & B^TU \\ \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & -R_2 & 0 \\ \star & \star & \star & \star & \star & -U \end{bmatrix} < 0, \tag{9}$$

$$\Psi_1 = \begin{bmatrix} X & K \\ \star & Z_1 \end{bmatrix} \geqslant 0, \tag{10}$$

$$\Psi_2 = \begin{bmatrix} Y & L \\ \star & Z_2 \end{bmatrix} \geqslant 0, \tag{11}$$

$$\Psi_3 = \begin{bmatrix} Y & M \\ \star & Z_2 \end{bmatrix} \geqslant 0, \tag{12}$$

$$\Psi_4 = \begin{bmatrix} D & N \\ \star & Z_3 \end{bmatrix} \geqslant 0, \tag{13}$$

$$\Psi_5 = \begin{bmatrix} X - Y & W \\ \star & Z_1 - Z_2 \end{bmatrix} \geqslant 0, \tag{14}$$

where

$$\begin{aligned}
\Psi_{11} &= PA + A^TP + Q_1 + R_1 + R_2 + K_1 + K_1^T + N_1 + N_1^T + h_1X_{11} + h_2Y_{11} + hD_{11},\\
\Psi_{12} &= W_1 - K_1 + L_1 + K_2^T + N_2^T + h_1X_{12} + h_2Y_{12} + hD_{12},\\
\Psi_{13} &= PB - L_1 + M_1 + K_3^T + N_3^T + h_1X_{13} + h_2Y_{13} + hD_{13},\\
\Psi_{22} &= -(1-\tau_1)(Q_1 - Q_2) + W_2 + W_2^T - K_2 - K_2^T + L_2 + L_2^T + h_1X_{22} + h_2Y_{22} + hD_{22},\\
\Psi_{23} &= -L_2 + M_2 + W_3^T - K_3^T + L_3^T + h_1X_{23} + h_2Y_{23} + hD_{23},\\
\Psi_{33} &= -(1-\tau)Q_2 - L_3 - L_3^T + M_3 + M_3^T + h_1X_{33} + h_2Y_{33} + hD_{33},\\
U &= h_1Z_1 + h_2Z_2 + hZ_3.
\end{aligned}$$

Proof. Choose a Lyapunov functional candidate to be:

$$\begin{aligned}
V(x(t)) &= V_1(x(t)) + V_2(x(t)) + V_3(x(t)),\\
V_1(x(t)) &= x^T(t)Px(t) + \int_{t-d_1(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-d(t)}^{t-d_1(t)} x^T(s)Q_2x(s)ds,\\
V_2(x(t)) &= \int_{t-h}^{t} x^T(s)R_1x(s)ds + \int_{t-h_1}^{t} x^T(s)R_2x(s)ds,\\
V_3(x(t)) &= \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta + \int_{-h}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_3\dot{x}(s)dsd\theta,
\end{aligned} \tag{15}$$

where $P>0, Q_1 \geqslant Q_2 \geqslant 0, R_1 \geqslant 0, R_2 \geqslant 0, Z_1 \geqslant Z_2 > 0$ and $Z_3 > 0$ are to be determined. From the Newton–Leibniz formula, the following equations are true for any matrices W, K, L, M and N with appropriate dimensions:

$$2\left[x^T(t)W_1 + x^T(t-d_1(t))W_2 + x^T(t-d(t))W_3\right]\left[x(t-d_1(t)) - x(t-h_1) - \int_{t-h_1}^{t-d_1(t)} \dot{x}(s)ds\right] = 0, \tag{16}$$

$$2\left[x^T(t)K_1 + x^T(t-d_1(t))K_2 + x^T(t-d(t))K_3\right]\left[x(t) - x(t-d_1(t)) - \int_{t-d_1(t)}^{t} \dot{x}(s)ds\right] = 0, \tag{17}$$

$$2\left[x^T(t)L_1 + x^T(t-d_1(t))L_2 + x^T(t-d(t))L_3\right]\left[x(t-d_1(t)) - x(t-d(t)) - \int_{t-d(t)}^{t-d_1(t)} \dot{x}(s)ds\right] = 0, \tag{18}$$

$$2\left[x^T(t)M_1 + x^T(t-d_1(t))M_2 + x^T(t-d(t))M_3\right]\left[x(t-d(t)) - x(t-h) - \int_{t-h}^{t-d(t)} \dot{x}(s)ds\right] = 0, \tag{19}$$

$$2\left[x^T(t)N_1 + x^T(t-d_1(t))N_2 + x^T(t-d(t))N_3\right]\left[x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(s)ds\right] = 0. \qquad \square \tag{20}$$

Moreover, similar to [23], for any appropriately dimensioned matrices $X = X^T \geqslant 0, Y = Y^T \geqslant 0$ and $D = D^T \geqslant 0$, the following equations hold:

$$0 = \int_{t-h_1}^{t} \eta^T(t)X\eta(t)ds - \int_{t-h_1}^{t} \eta^T(t)X\eta(t)ds = h_1\eta^T(t)X\eta(t) - \int_{t-h_1}^{t-d_1(t)} \eta^T(t)X\eta(t)ds - \int_{t-d_1(t)}^{t} \eta^T(t)X\eta(t)ds, \tag{21}$$

$$\begin{aligned} 0 &= \int_{t-h}^{t-h_1} \eta^T(t)Y\eta(t)ds - \int_{t-h}^{t-h_1} \eta^T(t)Y\eta(t)ds \\ &= h_2\eta^T(t)Y\eta(t) - \int_{t-h}^{t-d(t)} \eta^T(t)Y\eta(t)ds - \int_{t-d(t)}^{t-d_1(t)} \eta^T(t)Y\eta(t)ds - \int_{t-d_1(t)}^{t-h_1} \eta^T(t)Y\eta(t)ds, \end{aligned} \tag{22}$$

$$0 = \int_{t-h}^{t} \eta^T(t)D\eta(t)ds - \int_{t-h}^{t} \eta^T(t)D\eta(t)ds = h\eta^T(t)D\eta(t) - \int_{t-h}^{t} \eta^T(t)D\eta(t)ds, \tag{23}$$

where $\eta(t) = \left[x^T(t) \quad x^T(t-d_1(t)) \quad x^T(t-d(t))\right]^T$.

Calculating the derivative of $V(x(t))$ along the solutions of system (1)

$$\begin{aligned} \dot{V}_1(x(t)) &\leqslant 2x^T(t)PAx(t) + 2x^T(t)PBx(t-d(t))\dot{V}_2(x(t)) + x^T(t)Q_1x(t) - (1-\tau)x^T(t-d(t))Q_2x(t-d(t)) \\ &\quad - (1-\tau_1)x^T(t-d_1(t))(Q_1-Q_2)x(t-d_1(t)), \end{aligned} \tag{24}$$

$$\dot{V}_2(x(t)) = x^T(t)(R_1+R_2)x(t) - x^T(t-h)R_1x(t-h) - x^T(t-h_1)R_2x(t-h_1), \tag{25}$$

$$\dot{V}_3(x(t)) = \dot{x}^T(t)(h_1Z_1 + h_2Z_2 + hZ_3)\dot{x}(t) - \int_{t-h_1}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-h}^{t-h_1} \dot{x}^T(s)Z_2\dot{x}(s)ds - \int_{t-h}^{t} \dot{x}^T(s)Z_3\dot{x}(s)ds. \tag{26}$$

Combining (24)–(26) and adding the terms on the left side of (16)–(23) into the derivative of $V(x(t))$

$$\begin{aligned} \dot{V}(x(t)) &\leqslant \xi^T(t)\left[\Pi + \bar{A}^T(h_1Z_1 + h_2Z_2 + hZ_3)\bar{A}\right]\xi(t) - \int_{t-h_1}^{t-d_1(t)} \dot{x}^T(s)(Z_1-Z_2)\dot{x}(s)ds - \int_{t-d_1(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds \\ &\quad - \int_{t-h}^{t-d(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds - \int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds - \int_{t-h}^{t} \dot{x}^T(s)Z_3\dot{x}(s)ds - \int_{t-d_1(t)}^{t} \eta^T(t)X\eta(t)ds \\ &\quad - \int_{t-d(t)}^{t-d_1(t)} \eta^T(t)Y\eta(t)ds - \int_{t-h}^{t-d(t)} \eta^T(t)Y\eta(t)ds - \int_{t-h}^{t} \eta^T(t)D\eta(t)ds - \int_{t-h_1}^{t-d_1(t)} \eta^T(t)(X-Y)\eta(t)ds \\ &\leqslant \xi^T(t)\left[\Pi + \bar{A}^T(h_1Z_1 + h_2Z_2 + hZ_3)\bar{A}\right]\xi(t) - \int_{t-d_1(t)}^{t} \varsigma^T(t,s)\Psi_1\varsigma(t,s)ds - \int_{t-d(t)}^{t-d_1(t)} \varsigma^T(t,s)\Psi_2\varsigma(t,s)ds \\ &\quad - \int_{t-h}^{t-d(t)} \varsigma^T(t,s)\Psi_3\varsigma(t,s)ds - \int_{t-h}^{t} \varsigma^T(t,s)\Psi_4\varsigma(t,s)ds - \int_{t-h_1}^{t-d_1(t)} \varsigma^T(t,s)\Psi_5\varsigma(t,s)ds, \end{aligned} \tag{27}$$

where

$$\xi^T(t) = [x^T(t) \quad x^T(t-d_1(t)) \quad x^T(t-d(t)) \quad x^T(t-h) \quad x^T(t-h_1)], \varsigma(t,s) = \left[x^T(t) \quad x^T(t-d_1(t)) \quad x^T(t-d(t)) \quad \dot{x}^T(t)\right]^T,$$

$$\Pi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & -M_1-N_1 & -W_1 \\ \star & \Psi_{22} & \Psi_{23} & -M_2-N_2 & -W_2 \\ \star & \star & \Psi_{33} & -M_3-N_3 & -W_3 \\ \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & -R_2 \end{bmatrix},$$ $\Psi_{ij}, i,j = 1,2,3$, are defined in Theorem 1 and $\bar{A} = [A \quad 0 \quad B \quad 0 \quad 0]$.

Since $Z_i > 0, i = 1,2,3$, when $\Psi_i \geqslant 0, i = 1,2,\ldots,5$, then the last five parts in (27) are all less than 0. Thus, by Schur complements, we have

$$\dot{V}(x(t)) \leqslant \xi^T(t)[\Pi + \overline{A}^T(h_1Z_1 + h_2Z_2 + hZ_3)\overline{A}]\xi(t) < 0,$$

which is equivalent to (9) by Schur complements, then $\dot{V}(x(t)) < -\varepsilon\|x(t)\|^2$ for a sufficiently small $\varepsilon > 0$ and $x(t) \neq 0$, which ensures the asymptotic stability of system (1), see e.g. [26].

Remark 1. Theorem 1 presents a stability criterion for system (1) with two additive time-varying delay components. This criterion is derived by defining the new Lyapunov functional in (15), which makes full use of the information about $d_1(t)$ and $d_2(t)$.

Remark 2. In this paper, it is seen that $d_1(t), d(t) - d_1(t), h - d(t)$ are not simply enlarged as h_1, $h - h_1$ and h, respectively. Instead, the relationship that $d_1(t) + (h_1 - d_1(t)) = h_1$, $(h - d(t)) + (d(t) - d_1(t)) - (h_1 - d_1(t)) = h - h_1$ and $d(t) + (h - d(t)) = h$ is considered. And it is also worth mentioning that we did not ignore any useful terms in the calculation of the time derivative of $V(x(t))$.

In fact, Theorem 1 gives a criterion for system (1) with $d(t)$ satisfying (2) and (3). In many cases, the information of the derivative of delay is unknown. Regarding this circumstance, a rate-independent criterion for a delay only satisfying (2) is derived as follows by choosing $Q_1 = Q_2 = 0$ in Theorem 1.

Corollary 1. *For given scalars $h_1 > 0$ and $h_2 > 0$, the system (1) with delays $d_1(t)$ and $d_2(t)$ is asymptotically stable, if there exist matrices* $P > 0, R_1 \geqslant R_2 \geqslant 0, Z_1 \geqslant Z_2 > 0, Z_3 > 0, X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \star & X_{22} & X_{23} \\ \star & \star & X_{33} \end{bmatrix} \geqslant 0,\quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ \star & Y_{22} & Y_{23} \\ \star & \star & Y_{33} \end{bmatrix} \geqslant 0,\quad D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \star & D_{22} & D_{23} \\ \star & \star & D_{33} \end{bmatrix} \geqslant 0, K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}, L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$, *and* $W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$, *such that the following LMIs (28), and (10)–(14) hold:*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & -M_1 - N_1 & -W_1 & A^TU \\ \star & \widetilde{\Psi}_{22} & \Psi_{23} & -M_2 - N_2 & -W_2 & 0 \\ \star & \star & \widetilde{\Psi}_{33} & -M_3 - N_3 & -W_3 & B^TU \\ \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & -R_2 & 0 \\ \star & \star & \star & \star & \star & -U \end{bmatrix} < 0, \tag{28}$$

where

$$\widetilde{\Psi}_{22} = W_2 + W_2^T - K_2 - K_2^T + L_2 + L_2^T + h_1X_{22} + h_2Y_{22} + hD_{22},$$
$$\widetilde{\Psi}_{33} = -L_3 - L_3^T + M_3 + M_3^T + h_1X_{33} + h_2Y_{33} + hD_{33},$$

and other parameters are defined in Theorem 1.

Remark 3. It should be pointed out that the Corollary 1 derived in this paper is valid not only for the case where $d_1(t)$ and $d_2(t)$ are continuous and differential, but also for the case where $d_1(t)$ and $d_2(t)$ are continuous, but their derivatives do not exist.

The following result provides the feasible robust stability criterion for systems with the admissible uncertainty.

Theorem 2. *For given scalars $h_1 > 0, h_2 > 0, \tau_1$ and τ, the uncertain system (1) with delays $d_1(t)$ and $d_2(t)$ is robustly stable, if there exist matrices* $P > 0, Q_1 \geqslant Q_2 \geqslant 0, R_1 \geqslant R_2 \geqslant 0, Z_1 \geqslant Z_2 > 0, Z_3 > 0, X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \star & X_{22} & X_{23} \\ \star & \star & X_{33} \end{bmatrix} \geqslant 0, Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ \star & Y_{22} & Y_{23} \\ \star & \star & Y_{33} \end{bmatrix} \geqslant 0,\quad D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \star & D_{22} & D_{23} \\ \star & \star & D_{33} \end{bmatrix} \geqslant 0,\quad K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}, L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}, W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$, *and two positive scalars ε_i, $i = 1,2$, such that the following LMIs (29), and (10)–(14) hold:*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & -M_1-N_1 & -W_1 & A^TU & PH & 0 \\ \star & \Xi_{22} & \Xi_{23} & -M_2-N_2 & -W_5 & 0 & 0 & 0 \\ \star & \star & \Xi_{33} & -M_3-N_3 & -W_3 & B^TU & 0 & 0 \\ \star & \star & \star & -R_1 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -R_2 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -U & 0 & U^TH \\ \star & \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix} < 0, \tag{29}$$

where

$$\begin{aligned}
\Xi_{11} &= PA + A^TP + Q_1 + R_1 + R_2 + K_1 + K_1^T + N_1 + N_1^T + h_1X_{11} + h_2Y_{11} + hD_{11} + (\varepsilon_1+\varepsilon_2)E_a^TE_a, \\
\Xi_{12} &= W_1 - K_1 + L_1 + K_2^T + N_2^T + h_1X_{12} + h_2Y_{12} + hD_{12}, \\
\Xi_{13} &= PB - L_1 + M_1 + K_3^T + N_3^T + h_1X_{13} + h_2Y_{13} + hD_{13} + (\varepsilon_1+\varepsilon_2)E_a^TE_b, \\
\Xi_{22} &= -(1-\tau_1)(Q_1 - Q_2) + W_2 + W_2^T - K_2 - K_2^T + L_2 + L_2^T + h_1X_{22} + h_2Y_{22} + hD_{22}, \\
\Xi_{23} &= -L_2 + M_2 + W_3^T - K_3^T + L_3^T + h_1X_{23} + h_2Y_{23} + hD_{23}, \\
\Xi_{33} &= -(1-\tau)Q_2 - L_3 - L_3^T + M_3 + M_3^T + h_1X_{33} + h_2Y_{33} + hD_{33} + (\varepsilon_1+\varepsilon_2)E_b^TE_b, \\
U &= h_1Z_1 + h_2Z_2 + hZ_3.
\end{aligned}$$

Proof. By Lemma 1, the system is robustly, asymptotically stable if the following inequality holds:

$$\Psi + \Omega_1F(t)\Omega_2^T + \Omega_2F(t)\Omega_1^T + \Omega_3F(t)\Omega_4^T + \Omega_4F(t)\Omega_4^T < 0, \tag{30}$$

where

$$\begin{aligned}
\Omega_1 &= \begin{bmatrix} H^TP & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \Omega_2 = \begin{bmatrix} E_a & 0 & E_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
\Omega_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H^TU \end{bmatrix}^T, \Omega_4 = \begin{bmatrix} E_a & 0 & E_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\end{aligned}$$

By Lemma 2, Eq. (30) holds if the following inequality satisfies:

$$\begin{aligned}
&\Psi + \varepsilon_1^{-1}\Omega_1\Omega_1^T + \varepsilon_1\Omega_2\Omega_2^T + \varepsilon_2^{-1}\Omega_3\Omega_3^T + \varepsilon_2\Omega_4\Omega_4^T \equiv \Psi + \Omega \\
&= \Psi + \begin{bmatrix} \varepsilon_1^{-1}PHH^TP + (\varepsilon_1+\varepsilon_2)E_a^TE_a, & 0 & (\varepsilon_1+\varepsilon_2)E_a^TE_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & (\varepsilon_1+\varepsilon_2)E_b^TE_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \varepsilon_2^{-1}U^THH^TU \end{bmatrix} < 0,
\end{aligned} \tag{31}$$

where$\varepsilon_1 > 0, \varepsilon_2 > 0$.

Then, by Lemma 1, the inequality given in (31) is equivalent to the LMI (29). Thus, if the LMIs given in (29) and (10)–(14) hold, the system (1) is robust asymptotically stable. This completes the proof. □

By setting $Q_1 = Q_2 = 0$, Corollary 2 is established from Theorem 2.

Corollary 2. *For given scalars $h_1 > 0$ and $h_2 > 0$, the uncertain system (1) with delays $d_1(t)$ and $d_2(t)$ is robustly stable, if there exist matrices* $P > 0$, $R_1 \geqslant R_2 \geqslant 0$, $Z_1 \geqslant Z_2 > 0$, $Z_3 > 0, Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ \star & Y_{22} & Y_{23} \\ \star & \star & Y_{33} \end{bmatrix} \geqslant 0$, $D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ \star & D_{22} & D_{23} \\ \star & \star & D_{33} \end{bmatrix} \geqslant 0$, $K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}, L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}, W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$, *and two positive scalars ε_i, $i = 1,2$, such that the following LMIs (32), and (10)–(14) hold:*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & -M_1-N_1 & -W_1 & A^TU & PH & 0 \\ \star & \tilde{\Xi}_{22} & \Xi_{23} & -M_2-N_2 & -W_5 & 0 & 0 & 0 \\ \star & \star & \tilde{\Xi}_{33} & -M_3-N_3 & -W_3 & B^TU & 0 & 0 \\ \star & \star & \star & -R_1 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -R_2 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -U & 0 & U^TH \\ \star & \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix} < 0, \quad (32)$$

where

$$\tilde{\Xi}_{22} = W_2 + W_2^T - K_2 - K_2^T + L_2 + L_2^T + h_1X_{22} + h_2Y_{22} + hD_{22},$$
$$\tilde{\Xi}_{33} = -L_3 - L_3^T + M_3 + M_3^T + h_1X_{33} + h_2Y_{33} + hD_{33} + (\varepsilon_1+\varepsilon_2)E_b^TE_b,$$

and other parameters are defined in Theorem 2.

Remark 4. Though we only consider systems with two additive delay components, the results obtained in this paper can be readily extended to uncertain stochastic systems with multiple additive delay components, that is,

$$\mathrm{d}x(t) = \left[(A+\Delta A)x(t) + (B+\Delta B)x\left(t - \sum_{i=2}^{n} d_i(t)\right)\right]\mathrm{d}t + \left[(C+\Delta C)x(t) + (D+\Delta D)x\left(t - \sum_{i=2}^{n} d_i(t)\right)\right]\mathrm{d}\omega(t).$$

4. Numerical examples

In this section, we will give two examples showing the effectiveness of the results given here.

Example 1. Consider system (1) with the following parameters, borrowed from [4] and [9]:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \Delta A = 0, \quad \Delta B = 0.$$

Suppose we know that

$$\dot{d}_1(t) \leqslant 0.1, \quad \dot{d}_2(t) \leqslant 0.8.$$

Here, we let $d_1(t)$ represents the delay from sensor to controller and $d_2(t)$ represents the delay from controller to actuator. Our purpose is to find the upper bound h_1 of delay $d_1(t)$, or h_2 of $d_2(t)$, when the other is known, below which the system is asymptotically stable. By combining the two delay components together, some existing stability results can be applied to this system. The calculation results obtained by Theorem 1 in this paper, Theorem 1 in [21], Theorem 1 in [7], Theorem 2 in [15], Theorem 1 in [9], Theorem 3.2 in [11] and Theorem 1 in [4] for different cases are listed in Table 1, in which "–" means that the results are not applicable to the corresponding cases. It is clear that Theorem 1 gives much better results than those obtained by [22,21,7,4,9,15,11].

Example 2. Let us re-consider the time-delay system (1) with the following parameter:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad E_a = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad E_b = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad H = I.$$
$$\dot{d}_1(t) \leqslant 0.1, \quad \dot{d}_2(t) \leqslant 0.8.$$

Table 1
Calculated delay bounds for different cases.

	Delay bound of h_2 for given h_1				Delay bound of h_1 for given h_2		
	$h_1 = 1$	$h_1 = 1.1$	$h_1 = 1.2$	$h_1 = 1.5$	$h_2 = 0.3$	$h_2 = 0.4$	$h_2 = 0.5$
[11]	–	–	–	–	–	–	–
[15,9,4]	0.180	0.080	–	–	0.880	0.780	0.680
[7]	0.378	0.278	0.178	–	1.078	0.978	0.878
[21]	0.415	0.376	0.340	0.248	1.324	1.039	0.806
[22]	0.512	0.457	0.406	0.283	1.453	1.214	1.021
Theorem 1	0.872	0.772	0.672	0.371	1.572	1.472	1.372

Table 2
Calculated delay bounds for different cases.

	Delay bound of h_2 for given h_1			Delay bound of h_1 for given h_2		
	$h_1 = 0.1$	$h_1 = 0.2$	$h_1 = 0.5$	$h_2 = 0.1$	$h_2 = 0.2$	$h_2 = 0.3$
[15,1]	0.624	0.524	0.224	0.624	0.524	0.424
[7]	0.752	0.652	0.352	0.752	0.652	0.552
Theorem 2	1.771	1.671	1.372	1.770	1.672	1.571

The calculation results obtained by Theorem 2 in this paper, Corollary 1 in [7], Theorem 4 in [15] and Corollary 1 in [1] for different cases are listed in Table 2. It can be seen from the table that Theorem 2 in this paper yields the least conservative stability test than other single delay approaches, showing the advantage of the stability result in this paper.

5. Conclusions

This paper has investigated the stability problem for a class of uncertain systems with two successive delay components. By considering the relationship between the time-varying delay and its upper bound when calculating the upper bound of the derivative of Lyapunov functional, some improved stability criteria have been presented to guarantee the systems are robustly, asymptotically stable for all admissible parameter uncertainties. Numerical examples have also been used to demonstrate the usefulness of the main results and less conservativeness of the proposed method.

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